STATISTICAL HYPOTHESES and their TESTS

STATISTICAL HYPOTHESIS...

a supposition (assertion, conjecture) concerning the (unknown) distribution of RV.
This distribution is:

$$f = f(X, \lambda) \equiv f(X, \boldsymbol{\lambda})$$

$$\text{HYPOTHESIS} \quad \begin{cases} \text{parametric} & \begin{cases} \text{simple} \\ \text{composite} \end{cases} \\ \text{non-parametric} \end{cases}$$

SIMPLE PARAMETRIC HYPOTHESIS — e.g. "$N(0,1)$"
COMPOSITE PARAMETRIC HYPOTHESIS — e.g. "$N(0, \sigma); \quad \sigma = ?$"
NON-PARAMETRIC HYPOTHESIS — e.g. —
"the radioactive decay occurs in agreement with the Poisson distribution"

TO VERIFY A STATISTICAL HYPOTHESIS...

one must have a STATISTICAL SAMPLE:

$$X_1, X_2, \ldots, X_n \equiv \boldsymbol{X}$$

USUALLY WE TRY TO VERIFY THE NULL HYPOTHESIS H_0, WHICH IS A CERTAIN ,,ASSUMPTION" CONCERNING THE PARAMETER(S) OF THE DISTRIBUTION:

$$H_0 \equiv H_0 : \boldsymbol{\lambda} = \boldsymbol{\lambda}_0$$

VERSUS ANOTHER HYPOTHESIS — THE SO-CALLED *alternative hypothesis* H_1:

$$H_1 \equiv H_1 : \boldsymbol{\lambda} = \boldsymbol{\lambda}_1$$

IN ORDER TO VERIFY A HYPOTHESIS WE FORM A STATISTIC — A FUNCTION OF THE RV WHICH THE STATISTICAL SAMPLE IS COMPOSED OF

MORE PRECISELY — WE INSPECT WHAT IS THE PROBABILITY FOR THE GIVEN VALUE OF OUR STATISTIC TO OCCUR IF THE HYPOTHESIS IS A CORRECT ONE.

example:

Our property X follows the Gaussian: $N(m, 1)$ — $H_0 : (m = 0)$ $[H_1 : (m \neq 0)]$
random sample: $X_1, \ldots, X_{10}$; $\bar{x} = 1,01$
verification: $\bar{x}$ is the value of statistic $\bar{X}$, which follows another Gaussian

$$N(0, \sigma/\sqrt{n}) = N(0, 1/\sqrt{10})$$

we may standardise X in the usual way:

$$\bar{X} \to \frac{\bar{X} - 0}{\frac{1}{\sqrt{10}}} \equiv Z$$

$$\begin{aligned} P(|\bar{X}| \geq 1,01) &= P(|z| \geq \bar{x}\sqrt{10}) = P(|z| \geq 1,01\sqrt{10} \approx 3,05) \\ &= 2[1 - i(3,05)] = 0,002 \end{aligned}$$

if we insist on the correctness of the H_0 hypothesis the obtained result is very, very little plausible.
SIGNIFICANCE LEVEL (of the test) — the probability $P = \alpha$ of rejecting a true hypothesis H_0.
Usually we choose $\alpha = 0,01$ or $\alpha =$ **0,05**; (if the probability of obtaining a given value of the test statistic is $\leq \alpha$ we reject H_0).

another example:

OUR PROPERTY X FOLLOWS $N(m,1)$ — $H_0 : (m = 0)$ $[H_1 : (m \neq 0)]$
WE FORM A RANDOM SAMPLE: $X_1, \ldots, X_{16}$; $\bar{x} = 0,1$
verification: $\bar{x}$ is a given value of the statistic $\bar{X}$, which follows the distribution

$$N(0, \sigma/\sqrt{n}) = N(0, 1/4)$$

$$P(|\bar{X}| \geq 0,1) = P(|z| \geq 0,4) = 2[1 - i(0,4)] \approx 0,69$$

H_0 seems to be very sensible.

how do we proceed in order to TEST a statistical hypothesis ?

1. we define a TEST STATISTIC $T = T(\mathbf{X}, n)$
2. we choose the significance level α and the **CRITICAL REGION** (or *rejection region*) Z_{cr} of the values which may take on T, defined as

$$P(T \in Z_{cr}|H_0) = \alpha$$

3. we form a sample (size n), and we calculate the value of t_n of T; if

$$t = t_n \in Z_{cr}$$

   we reject H_0

N.B. The non-rejection (acceptance) of a hypothesis implies that the data do not give sufficient evidence to refute it. On the other hand, rejection implies that the SAMPLE EVIDENCE refutes it — or, putting it another way, there is a small probability of obtaining the sample information when, in fact, the hypothesis is true.

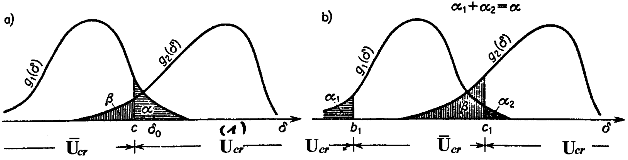
We risk to commit the two types of error:

- rejection of a true hypothesis —
  with the probability $= \alpha$ — the so-called type I error
- acceptance of a false hypothesis —
  with the probability $= \beta$

$$P(T \notin Z_{cr} | H_1) = \beta$$

— the so-called type II error

	H_0 is true	H_0 is false
Accept H_0	Correct decision	Type II error
Reject H_0	Type I error	Correct decision



With a fixed significance level α, we try to chose the test statistic T and the critical region Z_{cr} in such a way that the probability of the second type error — β — be as little as possible

A test, which for a given α minimises β is called the most powerful test for given H_0 versus a given alternative H_1. The **POWER of the test** is the quantity $1-\beta$, i.e. the probability of NOT rejecting the H_0 given that a specific alternative (H_1) is true.