POISSON DISTRIBUTION

The RV of discrete type — the number of outcomes occurring, for instance, during a given time (e.g. number of radioactive decays in a sample of radioactive material) t;

$$X = X_t = 0, 1, 2, \ldots$$

(also No of events in a given region of space — e.g. number of typing errors per page)

occurrences of Poisson distribution . . .

. . . are very (!) numerous. Just a few (from Wiki):

- telephone calls arriving during a (short) period of time;
- light quanta (photons) arriving at detecting system;
- number of mutations on a strand of DNA (per unit length);
- number of customers arriving at a counter;
- number of cars arriving at a traffic light;
- number of Losses/Claims;
- number of radioactive decays;
- requests for a particular document on a web server
- number of typing errors in a page of a draft
- . . . and many, many others.

BASIC PROPERTIES

1. $$W_i(t) \equiv \mathcal{P}(X_t = i) \quad i = 0, 1, 2, \ldots$$
2. Numbers of outcomes occurring in one time interval Δt are independent of each other, i.e. the number occurring in one time interval is independent of the number that occurs in any other disjoint time interval (Poisson process has no memory)
3. The probability that a single outcome will occur during a very short time interval Δt is PROPORTIONAL to the length of interval: $\mathcal{P}\left(X_{\Delta t} = 1\right)$
   $$\mathcal{P}\left(X_{\Delta t} = 1\right) \propto \Delta t$$
   or, more precisely,
   $$\lim_{\Delta t \to 0} \frac{W_1(\Delta t)}{\Delta t} = \lambda$$
4. The probability that more than one outcome will occur in such a short time interval is negligible
   $$\lim_{\Delta t \to 0} \frac{1 - W_0(\Delta t) - W_1(\Delta t)}{\Delta t} = 0$$

The formula . . .

. . . defining the Poisson distribution

$$\boxed{\mathcal{P}(X_t = k; \lambda) = \frac{\lambda^k}{k!} e^{-\lambda}}$$

The usual basic parameters of the Poisson distributions are:

$$E\{X\} = \lambda \quad VAR\{X\} = \lambda \quad \sigma(X) = \sqrt{\lambda}.$$

Note: The mean and the variance of the Poisson distribution are equal to each other and to the unique distribution parameter λ.
N.B. – try to calculate it yourself:

$$E(X) = \sum_{k=0}^{\infty} k \cdot \frac{\lambda^k}{k!} e^{-\lambda} = ?$$

(the same – a bit harder – for σ^2).

The formula – derivation – . . .

$$\sum_{k=0}^{\infty} k \cdot \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \boxed{k - 1 = n} = e^{-\lambda} \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{(n)!} =?$$

The Poisson distribution...

...can be viewed as a limiting case of the binomial distribution W_k^n for a large number of trials $n \gg 1$ and a very small probability of single success (the parameter p is close to zero; $q = 1 - p$ is close to unity). The product $n\,p$ is the Poisson distribution parameter λ.

outline of verification:

$$\begin{aligned} W_k^n(p) &= \begin{pmatrix} n \\ k \end{pmatrix} p^k q^{n-k} = \frac{n(n-1)\ldots(n-k+1)}{k!} p^k (1-p)^{n-k} \\ &= \boxed{p = \frac{\lambda}{n}} \quad \frac{n(n-1)\ldots(n-k+1)}{k!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\ &= 1\left(1 - \frac{1}{n}\right) \ldots \left(1 - \frac{k-1}{n}\right) \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-k} \end{aligned}$$

As $n \to \infty$ while k and λ remain constant the k first factors and the last one tend to unity . From the definition of the number e:

$$\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}.$$

Example: (Rosenkrantz)

Hence, under the given limiting conditions,

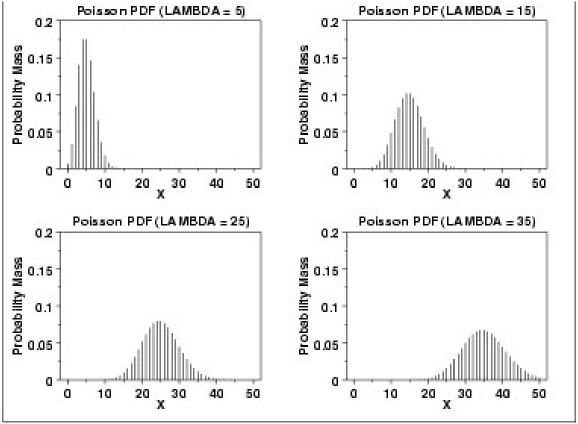
$$\boxed{W_k^n(p) \to \frac{e^{-\lambda}\,\lambda^k}{k!}} \qquad k = 0, 1, 2, \ldots$$

Example:
The probability of getting leukemia is $p = 0.000248$. Using the approximation Bernoulli-to-Poisson find the $\mathcal{P}$ of eight or more leukemia cases in a population of size $n = 7076$.
$np = 7076 \times 0.000248 = 1.75 = \lambda$.

$$\mathcal{P}(X \leq 7) = \sum_{0 \leq x \leq 7} e^{-1.75} \frac{1.75^x}{x!} = 0.999518.$$

hence $\mathcal{P}(X \geq 8) \approx 1 - 0.999518 = 0.000482$.

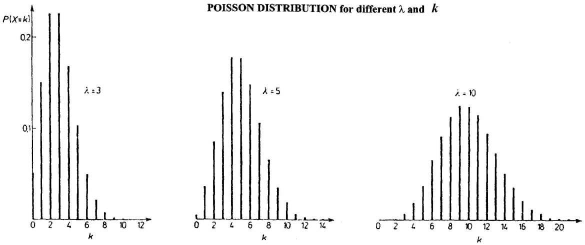
here come few graphs of Poisson distribution for different λ values:



NIST/SEMATECH e-Handbook of Statistical Methods

http://www.itl.nist.gov; Jan 5th 2012

again:


POISSON DISTRIBUTION for different λ and k
P(X=k)
0.2
0.1
λ=3
λ=5
λ=10
k
k
k