1D AND 2D RANDOM VARIABLES

1. An absent-minded professor has a bunch of 5 keys. One of these opens the door to his apartment. His usual strategy is: he selects a key at random and tries it in the lock. If it does not work he replaces the key and selects another one from the whole bunch. Let X denote the number of attempts the prof makes. What is the probability function of X – $f(x)$?

2. X is a discrete random variable (with an infinite but countable # of values) with p. f.

$$\mathcal{P}(X = x) = f(x) = \left(\frac{1}{2}\right)^x; \qquad x = 1, 2, \ldots.$$

   Find the probability that X is even.

3. The deck of cards can dichotomized into hearts (13) and all other cards (39). What is the p of getting a heart on a single draw (1/4). What is probability q of getting getting a spade OR club OR diamond (3/4). When 7 cards are sampled **with replacement** what is the $\mathcal{P}$ of: a) getting no hearts at all?; b) getting 4 hearts? What is the $\mathcal{P}$ of getting 2 hearts out of the first four draws and then 2 hearts out of the next 3? Is the last result more or less probable than '4 hearts out of 7'? Why?

   Hint: the second $\mathcal{P}$ is less because the scheme restricts the # of ways the total of 4 can be arranged.

4. The most common application of the binomial theorem in industrial work is in in lot-by-lot acceptance inspection. If there are a certain # of defectives the lot will be rejected.

   It's natural to wish to find the $\mathcal{P}$ that the lot is acceptable even though a certain # of defectives are observed. Let p be the fraction of defectives. We assume that the size of the sample is **small** compared to the lot size. This keeps the p almost constant.

   Let's now choose a sample from a lot where 10% of the items are defective. What is the $\mathcal{P}$ of observing 0, 1 or 2 defectives in the sample. Assume the sample size 18.

5. The $\mathcal{P}$ of hitting a target on a shot is 2/3. A person fires 8 shots; the total # of hits is X. Find:

$$\mathcal{P}(X = 3); \quad \mathcal{P}(1 < X \leq 6); \quad \mathcal{P}(X > 3).$$

Joint distribution

6. Two random variables (X, Y) have a dstr which is uniform over the unit circle with the center at $(0, 0)$. Find the joint probability density function $f(x, y)$ and the marginal distributions $g(x)$ and $h(y)$ Are X and Y independent?

   Hint: we have $f(x, y) = C$. From the normalization condition

$$\int_{\text{circle}} f(x, y)\, dxdy = 1 \quad \rightarrow \quad C = 1/\pi.$$

7. The same as in 6) but for

$$f(x,y) = x + y; \qquad 0 < x < 1; \quad 0 < y < 1; \quad f(x,y) = 0 \text{ otherwise}$$

8. The same as in 6) but for

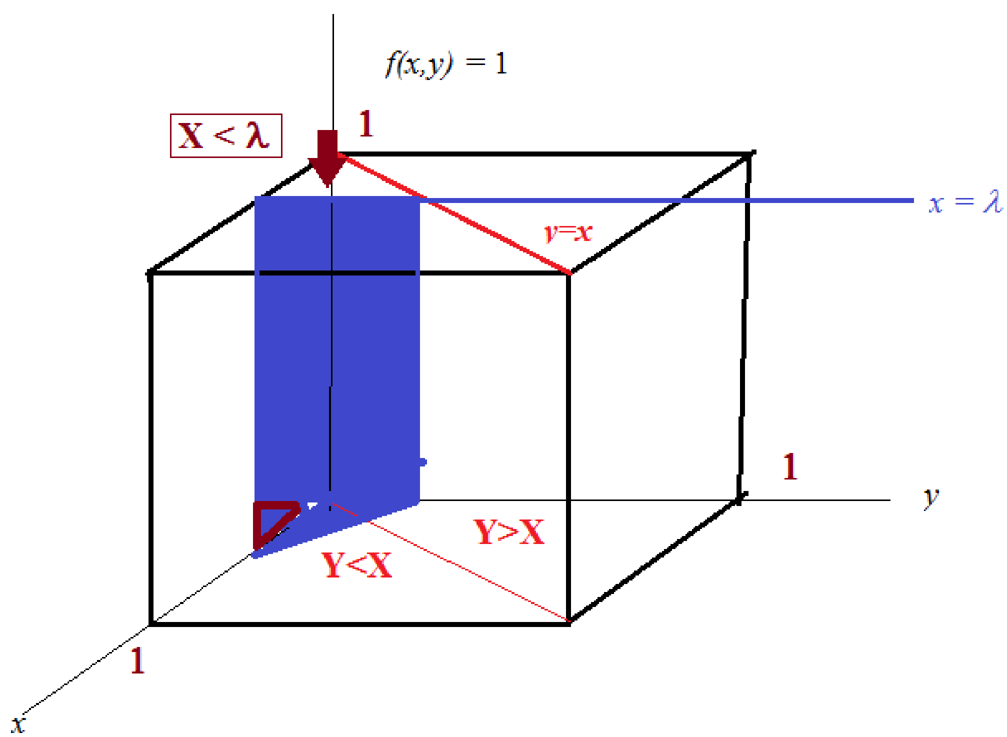
$$f(x,y) = e^{-(x+y)}; \qquad 0 < x < \infty; \quad 0 < y < \infty; \quad f(x,y) - 0 \text{ otherwise}$$

9. A two-D RV has the joint probability density function

$$f(x,y) = 1; \qquad 0 < x < 1; \quad 0 < y < 1; \quad f(x,y) = 0 \text{ otherwise}$$

Find the *conditional cumulative distribution function*

$$F(\lambda|X > Y) = \mathcal{P}(X \le \lambda | X > Y) = \frac{\mathcal{P}(X \le \lambda \text{ and } X > Y)}{\mathcal{P}(X > Y)}.$$



Hint: consider the (crude) graph of the situation. The constant $f(x,y) = 1$ hovers over the $(0,0),(0,1),(1,1),(1,0)$ square in the xy plane forming a cube withe the volume equal to 1. The $\mathcal{P}$ of $(X \le \lambda \text{ and } X > Y)$ is the volume behind the blue screen - it's ...??

10. Industrial application. Let a random variable Y represent the diameter of a shaft and a random variable X represent the inside diameter of the housing that is intended to support the shaft. By design the diameter of the shaft is 99.5 units and the inside diameter of the housing – 100 units. However, the manufacturing process of both shaft and housing is imperfect. In fact Y is uniformly distributed over the interval $(98.5, 100.5)$ and X is uniformly distributed over the interval $(99, 101)$. What is the $\mathcal{P}$ that a particular shaft

can be successfully paired with a particular housing, i.e. we have $X - h < Y < X$ for some small positive h. Use $h = 0.5$ for computations.

Hint: the distribution density function of X – $f(x) = \frac{1}{2}$ over $[99, 101]$ interval; analogously $g(y) = \frac{1}{2}$ over $[98.5, 100.5]$ interval.