PARAMETRIC STATISTICAL TESTS
testing the equality of
EXPECTED VALUES of 2 RV's

Given RV X follows — in two (different) populations distributions

$$N(\mu_1, \sigma_1), \quad N(\mu_2, \sigma_2); \;\; \sigma_1 = \sigma_2 = \sigma \;\; \text{(unknown)}$$

(these do not have to be The Normal distributions!)
HYPOTHESIS TO BE VERIFIED: $H_0 : \mu_1 = \mu_2 \equiv \mu$

We form two samples (each from every population):

$$\text{sample 1:} \quad X_1, X_2, \ldots, X_{n_1} \quad \text{size } n_1$$

$$\text{sample 2:} \quad Y_1, Y_2, \ldots, Y_{n_2} \quad \text{size } n_2$$

and we construct the "usual" estimators:

$$\bar{X} = \frac{1}{n_1} \sum_{k=1}^{n_1} X_k \quad \bar{Y} = \frac{1}{n_2} \sum_{k=1}^{n_2} Y_k$$

and

$$S_X^2 = \frac{1}{n_1} \sum_{k=1}^{n_1} (X_k - \bar{X})^2 \quad S_Y^2 = \frac{1}{n_2} \sum_{k=1}^{n_2} (Y_k - \bar{Y})^2$$

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$$\bar{X} \hat{=} N\left(\mu, \frac{\sigma}{\sqrt{n_1}}\right) \qquad \bar{Y} \hat{=} N\left(\mu, \frac{\sigma}{\sqrt{n_2}}\right)$$

so their difference has the distribution:

$$\bar{X} - \bar{Y} \hat{=} N\left(0, \sigma\sqrt{\frac{n_1 + n_2}{n_1\, n_2}}\right)$$

as we have

$$\sigma^2(\bar{X} - \bar{Y}) = \sigma^2(\bar{X}) + \sigma^2(\bar{Y}) = \frac{\sigma^2}{n_1} + \frac{\sigma^2}{n_2} = \sigma^2 \frac{n_1 + n_2}{n_1 n_2}$$

Now — we have to transform the normal distribution into the Student's distribution (we don't know σ!) — following the same scheme we employed in the case of the confidence intervals:

$$N \hat{=} U \equiv \frac{\bar{X} - \mu}{\sigma}\sqrt{n} \rightarrow \frac{\bar{X} - \mu}{S}\sqrt{n-1} \equiv t \hat{=} St$$

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1. we divide U by the RV $\sqrt{n}S/\sigma$, whose square:

$$\frac{nS^2}{\sigma^2} \quad \text{follows the distribution} \quad \chi^2_{n-1}$$

2. we multiply the resulting quantity by the square–root of the number of degrees of freedom $\sqrt{n-1}$
   For the case of 2 independent samples the variable analogous to nS^2/σ^2 is: $\dfrac{n_1 S_X^2 + n_2 S_Y^2}{\sigma^2}$ and the number of degrees of freedom is equal to: $n_1 - 1 + n_2 - 1$.

If so, the standardised variable

$$\frac{\bar{X} - \bar{Y}}{\sigma\sqrt{\dfrac{n_1 + n_2}{n_1 n_2}}}$$

must be divided by $\sqrt{n_1 S_X^2 + n_2 S_Y^2}/\sigma$ and multiplied by $\sqrt{n_1 + n_2 - 2}$.

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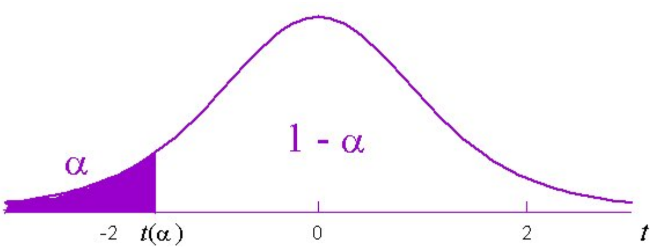
$$\Delta = \frac{\bar{X} - \bar{Y}}{\sqrt{n_1 S_X^2 + n_2 S_Y^2}} \times \sqrt{\frac{n_1 n_2 (n_1 + n_2 - 2)}{n_1 + n_2}}$$

follows Student's distribution with $n_1 + n_2 - 2$ degrees of freedom.
Testing $H_0 : \mu_1 = \mu_2$ consists in verifying whether the value of $\Delta(\text{calculated from the 2 samples}) \in U_{cr}(\alpha)$.
The critical (rejection) region $U_{cr}(\alpha, H_1)$ is defined by appropriate quantiles of the Student distribution.

Testing $H_0 : \mu_1 = \mu_2$ – critical (rejection) region $U_{cr}(\alpha, H_1)$ may be: $(nn = n_1 + n_2 - 2)$

1. $H_1 : \ \mu_1 < \mu_2; \quad U_{cr} = \Big(-\infty, -t(1-\alpha, nn)\Big]$



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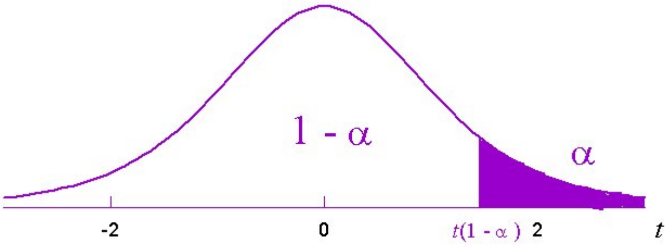
2. $H_1 : \ \mu_1 > \mu_2; \quad U_{cr} = \Big[+t(1-\alpha, nn), \infty \Big)$



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3. $H_1 \; \mu_1 \neq \mu_2$;

$$U_{cr} = \Big(-\infty, -t(1-\tfrac{\alpha}{2}, nn) \Big) \bigcup \Big(+t(1-\tfrac{\alpha}{2}, nn), \infty \Big)$$