PROBABILITIES, RV BASICS

1. [4 points] After finishing his book, the author goes over the text and finds m_1 mistakes. After that, he proofreads it another time and finds m_2 mistakes ($m_2 < m_1$). How many mistakes can he expect to still be in the book?

   Hint: we assume the probability p of spotting a mistake to be the same during 1st and 2nd read.

   # of mistakes original – M
   # of mistakes after the 1st read $M - m_1$; after second: $L = M - m_1 - m_2$.

   probability of spotting a mistake: $p = \frac{m_1}{M}$; hence $m_2 = p(M - m_1) = \ldots = m_1\left(1 - \frac{m_1}{M}\right)$.

   Thus $\boxed{M = \frac{m_1^2}{m_1 - m_2}}$ and $L = M - m_1 - m_2 = \ldots = \frac{m_2^2}{m_1 - m_2}$;
   $p = m_1/M = 1 - \frac{m_2}{m_1}$.

   Try to illustrate the problem with $m_1 = 30$ and $m_2 = 12$.

2. [4 points] In a poker hand find the probability of holding (a) 3 aces; (b) 4 aces and 1 club.

   (a)
   $$\binom{4}{3} \times \binom{48}{2} / \binom{52}{5}$$
   (b)
   $$12/\binom{52}{5}$$

3. [3 points] Probability that a regularly scheduled flight departs on time is $\mathcal{P}(D) = 0.83$; the probability that it arrives on time is $\mathcal{P}(A) = 0.82$. The probability that it departs and arrives on time is $\mathcal{P}(AD) = 0.78$.

   (a) are the events: 'departure on time' , and, 'arrival on time' dependent or independent? Find the probability that a plane will

   (b) arrive on time given that it departed on time

   (c) departed on time, given that it arrived on time.

   (a) dependent: $P(AD) \neq P(A)P(D)$

   (b) & (c) $P(AD) = P(A|D)P(D) = P(D|A)P(A)$

4. [4 points] Consider a deck of six cards marked 2, 4, 6, 8, 10 and 12. Two of these cards are picked at random without replacement; let W = the sum of numbers picked.

   (a) Compute the probability function of W (fill-in the probability table):

w_i	6	8	10	12	14	16	18	20	22
p_i	1/15	1/15	2/15	2/15	3/15	2/15	2/15	1/15	1/15

   Hint: construct an auxiliary table with numbers 2, 4, 6, 8, 10 and 12 as the headers of the columns and rows. Calculate the sums at the intersections.

   (b) fill in the cumulative distribution table: $F(w_i) = \mathcal{P}(W \le w_i)$:

w_i	6	8	10	12	14	16	18	20	22
$F(w_i)$	1/15	2/15	4/15	6/15	9/15	11/15	13/15	14/15	1

   and sketch (roughly) the 2 graphs. What is the probability of having the sum less than 18 and greater than or equal to 14? $\mathcal{P}(14 \le X < 18) = F(16) - F(12) = 1/3$

5. [4 points] Compute the expected value and the variance of the random variable W from the problem 4.

   $E(W) = \sum_i p_i w_i = 7$; (also from the symmetry)
   $VAR(W) = \sum_i p_i(w_i - 14)^2 = 2 \times \left[(6-14)^2 \cdot \frac{1}{15} + (8-14)^2 \cdot \frac{1}{15} + (10-14)^2 \cdot \frac{2}{15} + (12-14)^2 \cdot \frac{2}{15}\right] = \ldots = 56/3.$

   If you fail to construct the probability table calculate the expected value and the variance of the random variable V from the table below [3 points]

v_i	-1	0	1	3
p_i	0.2	0.3	0.3	0.2

$$E(V) = \sum_i p_i w_i = 0.7; \quad VAR(W) = \sum_i p_i (v_i - 0.7)^2 = 1.81$$

6. [4 points]The probability density function of the random variable X is given by:

$$f(x) = \begin{cases} 0 & \text{for} \quad -\infty < x < 0 \\ \cos x & \text{for} \quad 0 \leq x \leq \frac{1}{2}\pi \\ 0 & \text{for} \quad \frac{1}{2}\pi < x < \infty \end{cases}$$

Find the cumulative distribution $F(x)$ and calculate $\mathcal{P}(1/6\pi < x < 1/3\pi)$.

Sketch the graphs of probability density function and cumulative distribution function and read out the calculated $\mathcal{P}$ from the graphs.

$$F(x) = \int_{-\infty}^{x} f(s)\, ds = \int_{-\infty}^{x} \cos(s)\, ds = \sin x$$

$\mathcal{P}(1/6\pi < x < 1/3\pi) = F(1/3\pi) - F(1/6\pi) = \sin(1/3\pi) - \sin(1/3\pi) = \sqrt{3}/2 - 1/2 \approx 0.366.$

7. [3 points] A multiple-choice quiz has 10 questions each with four alternatives. A passing score is 5 or more correct. If a student attempts to guess the answer to each question what is the probability that she/he passes?

the probability of having five correct answers is

$$W_5^{10} = \binom{10}{5} \left(\frac{1}{4}\right)^5 \left(\frac{3}{4}\right)^5 \approx 0.0162$$

but the overall probability will be a sum

$$S = W_5^{10} + W_6^{10} + W_7^{10} + W_8^{10} + W_9^{10} + W_{10}^{10} === 1 - \mathcal{P}(X \leq 4) \ldots \text{tables} = 1 - 0.922 \approx 0.078.$$

8. [4 points]The average number of radioactive particles passing a counter during 1 millisecond in a laboratory experiment is 6.

What is the probability that:

a) exactly 5 particles enter the counter in a given millisecond?

$\mathcal{P}(X = 5) = \mathcal{P}(X \leq 5) - \mathcal{P}(X \leq 4) = \ldots \text{tables} \ldots 0.4457 - 0.2851 = 0.1606$

b) at least 5 particles enter the counter in a given millisecond?

it means that the number must be greater than 4.

from tables: $\mathcal{P}(X \leq 4) = 0.2851$; hence $\mathcal{P} = 1 - 0.2851 = 0.7149$

c) $\mathcal{P}(X \leq 5) = F(5) = 0.446$

9. [4 points] An electrical firm manufactures (traditional) light bulbs that have a length of life that is normally distributed with mean equal to 1000 hours and a standard deviation of 50 hours. Find the probability that a bulb burns between 985 and 1030 hours.

$$z_1 = \frac{985 - 1000}{50} = -0.3; \quad z_2 = \frac{1030 - 1000}{50} = 0.6$$

$\mathcal{P}(985 < X < 1030) = \mathcal{P}(-0.3 < Z < 0.6) = \mathcal{P}(Z < 0.6) - \mathcal{P}(Z < -0.3) = \Phi(0.6) - \Phi(-0.3) = 0.72575 - 0.38209 = 0.34366$