
RANDOM VARIABLE - BASIC CONCEPTS

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1. A continuous random variable has a cummulative distribution function defined as follows:

$$F(x) = \begin{cases} 0 & \text{for} \quad x \le 1, \\ 2\left(1 - \dfrac{1}{x}\right) & \text{for} \quad 1 < x \le a \\ 1 & \text{for} \quad x > a \end{cases} .$$

Find the value of the a constant. Find the probability density function of the X variable: $f(x)$. Calculate the probability

$$\mathcal{P}\,(-1 < X < 1.5).$$

For the end of the interval the distribution must be equal to $F(a) = 1$. Thus

$$2(1 - 1/a) = 1; \quad \text{hence} \quad a = 2.$$

$$\mathcal{P}\,(-1 < X < 1.5) = F(1.5) - F(-1) = F(1.5) = 2/3.$$

Joint distribution

2. Two random variables (X, Y) have a probability density distribution $f(x, y)$ which is uniform over the unit circle with the center at $(0, 0)$. Find the joint probability density function $f(x, y)$ and the marginal distributions $g(x)$ and $h(y)$ Are X and Y independent?

We have $f(x, y) = C$. From the normalization condition

$$\int_{\text{circle}} f(x, y)\, dxdy = 1 \quad \to \quad C = 1/\pi.$$

For a fixed $x = x_0$ the other variable $-\sqrt{1 - x_0^2} < y < \sqrt{1 - x_0^2}$; thus

$$g(x_0) = \int_{x=x_0} f(x, y)\, dy = \int_{-\sqrt{1-x_0^2}}^{\sqrt{1-x_0^2}} f(x, y)\, dy = 1/\pi \int_{-\sqrt{1-x_0^2}}^{\sqrt{1-x_0^2}} dy = \frac{2\sqrt{1 - x_0^2}}{\pi}.$$

In an analogous fashion

$$h(y_0) = \frac{2\sqrt{1 - y_0^2}}{\pi}.$$

The two variables are dependent since $f(x, y) \ne h(x) \times g(y)$.

3. The same as in 2) but for

$$f(x, y) = x + y; \qquad 0 < x < 1; \quad 0 < y < 1; \quad f(x, y) = 0 \text{ otherwise}$$

the marginal distributions are: $g(x) = 1/2 + y$ and $h(y) = 1/2 + x$

4. The same as in 2) but for

$$f(x, y) = e^{-(x+y)}; \qquad 0 < x < \infty; \quad 0 < y < \infty; \quad f(x, y) - 0 \text{ otherwise}$$

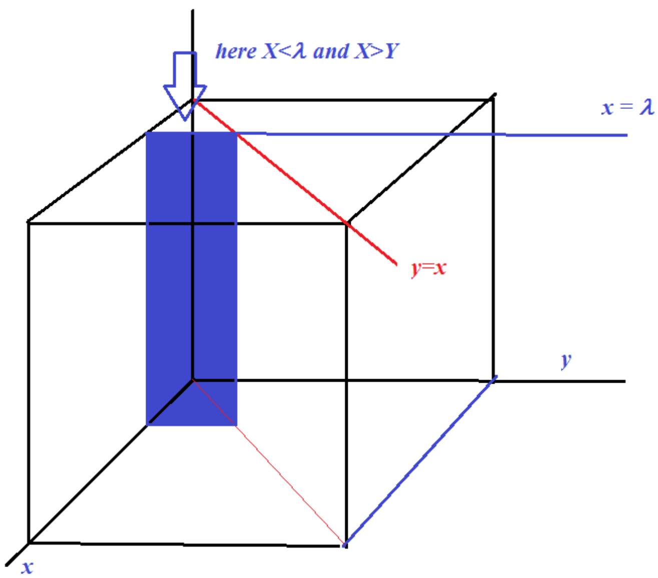
the marginal distributions are: $g(x) = e^{-x}$ and $h(y) = e^{-y}$ $\quad f(x, y) = g(x) \times h(y)$;independent

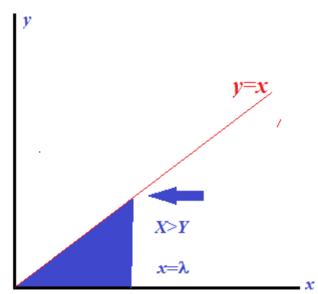
5. A two-D RV has the joint probability density function

$$f(x, y) = 1; \qquad 0 < x < 1; \quad 0 < y < 1; \quad f(x, y) = 0 \text{ otherwise}$$

Find the *conditional cumulative distribution function*

$$F(\lambda|X > Y) = \mathcal{P}(X \leq \lambda|X > Y) = \frac{\mathcal{P}(X \leq \lambda \text{ and } X > Y)}{\mathcal{P}(X > Y)}.$$





Hint: consider the (crude) graph of the situation. The constant $f(x, y) = 1$ hovers over the $(0, 0), (0, 1), (1, 1), (1, 0)$ square in the xy plane forming a cube with the volume equal to 1. The $\mathcal{P}$ of $(X \leq \lambda$ and $X > Y)$ is the volume behind the blue screen - it's a prism with a unit height whose base is a right triangle of side equal to λ. Thus the $\mathcal{P}(X \leq \lambda$ and $X > Y) = \ldots = \lambda^2/2$. On the other hand $\mathcal{P}(X > Y)$ (in the whole square) is obviously equal to $1/2$. Hence ...

6. Industrial application. Let a random variable Y represent the diameter of a shaft and a random variable X represent the inside diameter of the housing that is intended to support the shaft. By design the diameter of the shaft is 99.5 units and the inside diameter of the housing – 100 units. However, the manufacturing process of both shaft and housing is imperfect. In fact Y is uniformly distributed over the interval $(98.5, 100.5)$ and X is uniformly distributed over the interval $(99, 101)$. What is the $\mathcal{P}$ that a particular shaft can be successfully paired with a particular housing, i.e. we have $X - h < Y < X$ for some small positive h. Use $h = 0.5$ for computations.

Hint: the distribution density function of X – $f(x) = \frac{1}{2}$ over $[99, 101]$ interval; analogously $g(y) = \frac{1}{2}$ over $[98.5, 100.5]$ interval.

$h = 0.5$ – it's 25% of the $(99, 101)$ interval.

In the interval $99 < X < 100.5$ we have

$$\mathcal{P}(X - \frac{1}{2} < Y < X) = \int_{x-1/2}^{x} \frac{1}{2}\, dy = \frac{1}{4}$$

.

However, in the $100.5 < X < 101$ interval

$$\mathcal{P}(x - \frac{1}{2} < Y < x) = \int_{x-1/2}^{100.5} g(y)\, dy = \int_{x-1/2}^{100.5} \frac{1}{2}\, dy = \frac{1}{2}(100.5 - (x - 1/2)).$$

Finally

$$\mathcal{P}(X-h<Y<X)=\int_{\infty}^{\infty}\mathcal{P}(x-h<Y<x|X=x)\,f(x)dx=\frac{1}{2}\int_{99}^{101}\mathcal{P}(x-\frac{1}{2}<Y<x)\,dx$$
$$=\int_{99}^{100.5}\frac{1}{2}\cdot\frac{1}{4}\,dx+\int_{100.5}^{101}\frac{1}{2}(100.5-(x-1/2))\,dx$$
$$=\ldots=1/4$$

7. Suppose we have a group of 8 students: 3 Poles, 2 Spaniards and 3 Turks. We form a committee selecting – at random – 2 persons. Let X denote the number of Poles and Y – the number of Spaniards.

   (a) Prove that the joint-probability function $f(x,y)$ has the form:

   $$f(x=i,y=k;\,i,k=0,1,2)=\frac{\binom{3}{x}\binom{2}{y}\binom{3}{2-x-y}}{\binom{8}{2}}$$

   for $x=0,1,2$; $y=0,1,2$; $0\le x+y\le 2$.

   (b) Construct a table of all probabilities and inspect it carefully; check whether the sum of all $f(x,y)$ is equal to 1. Check whether the marginal probabilities (row/column totals) all sum up to unity.

$f(x,y)$	x 0	1	2	*Row Totals*
y 0	$\frac{??}{28}$	$\frac{??}{28}$	$\frac{??}{28}$	$\frac{??}{28}$
1	$\frac{??}{28}$	$\frac{??}{28}$		$\frac{??}{28}$
2	$\frac{??}{28}$			$\frac{??}{28}$
Column Totals	$\frac{}{28}$	$\frac{}{28}$	$\frac{??}{28}$	1

   (c) Find

   - Expected values of X and Y.
     ($E(X)=\mu_X=\frac{21}{28}$; $E(Y)=\mu_Y=\frac{14}{28}$).
   - Expected value of the product XY (6/28).
   - The covariance $COV(X,Y)=E(XY)-\mu_X\mu_Y$ (-9/56).