Stany stacjonarne w dwuwymiarowych układach zaburzanych szumami Lévy'ego

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Motivation: Boltzmann-Gibbs distribution

In the equilibrium

$$P(\text{state}) \propto \exp\left[-\frac{E}{k_B T}\right],$$

T – system temperature, E – energy of the state.

For an **overdamped** particle the Langevin equation is

$$\frac{dx}{dt} = -V'(x) + \sqrt{2k_B T}\xi(t).$$

Particle's energy is

$$E = V(x)$$

and the stationary distribution

$$P(x) \propto \exp\left[-\frac{V(x)}{k_B T}\right]$$

is fully determined by the potential $V(x)$.

Motivation & Outlook

Motivation

- Examination of stationary states for more general noises.

Road map of presentation

- basic definitions:
  - 1D α-stable noises,
  - 2D α-stable noises.
- stationary states for 1D and 2D systems.

Try to understand

- role of increasing spatial dimensionality,
- universalities of noise driven systems.

Take home message

2D α-stable noises differs from their 1D analogs but systems driven by 2D α stable noises display universal properties.

Noise in 1D

A random variable is X is stable if

$$AX^{(1)} + BX^{(2)} \stackrel{\mathrm{d}}{=} CX + D,$$

where $X^{(1)}$ and $X^{(2)}$ are independent copies of X, $\stackrel{\mathrm{d}}{=}$ denotes equality in distributions. The random variable X is called strictly stable if $D = 0$. The random variable X is symmetric stable if it is stable and

$$\mathrm{Prob}\{X\} = \mathrm{Prob}\{-X\}.$$

The random variable is α-stable if $C = (A^\alpha + B^\alpha)^{1/\alpha}$ where $0 < \alpha \leqslant 2$.

The characteristic function of α-stable densities is

$$\phi(k) = \mathbb{E}\left[e^{ikX}\right] = \begin{cases} \exp\left[-\sigma^\alpha |k|^\alpha \left(1 - i\beta \mathrm{sign} k \tan \frac{\pi\alpha}{2}\right) + i\mu k\right] \\ \qquad \text{if } \alpha \neq 1, \\ \exp\left[-\sigma |k| \left(1 + i\beta \frac{2}{\pi} \mathrm{sign} k \ln |k|\right) + i\mu k\right] \\ \qquad \text{if } \alpha = 1, \end{cases}$$

where $\alpha \in (0, 2]$, $\beta \in [-1, 1]$, $\sigma > 0$ and $\mu \in \mathbb{R}$.

Characteristic function

$$\phi(k) = \begin{cases} \exp\left[-\sigma^\alpha |k|^\alpha \left(1 - i\beta \operatorname{sign} k \tan\frac{\pi\alpha}{2}\right) + i\mu k\right], & \text{for } \alpha \neq 1, \\ \exp\left[-\sigma|k| \left(1 + i\beta\frac{2}{\pi} \operatorname{sign} k \ln|k|\right) + i\mu k\right], & \text{for } \alpha = 1, \end{cases}$$

- asymptotic behavior $P(x) \propto |x|^{-(\alpha+1)}$ $(\alpha < 2)$,
- Normal distribution $(\alpha = 2, \beta = 0)$

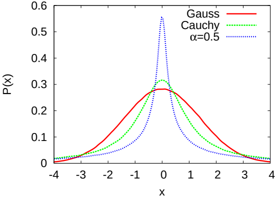
$$\frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right],$$

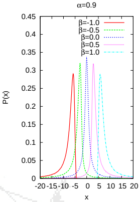
- Cauchy distribution $(\alpha = 1, \beta = 0)$

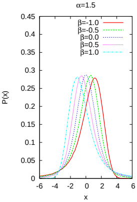
$$\frac{\sigma}{\pi} \frac{1}{(x-\mu)^2 + \sigma^2},$$

- Lévy-Smirnoff distribution (fully asymmetric, $\alpha = \frac{1}{2}, \beta = 1$)

$$\left(\frac{\sigma}{2\pi}\right)^{\frac{1}{2}} (x-\mu)^{-\frac{3}{2}} \exp\left[-\frac{\sigma}{2(x-\mu)}\right].$$







Analogously like in 1D: Random vector $\mathbf{X} = (X_1, \dots, X_d)$ is said to be a stable random vector in $\mathbb{R}^d$ if for any positive numbers A and B, there is a positive number C and a vector $\mathbf{D}$ such that

$$A\mathbf{X}^{(1)} + B\mathbf{X}^{(2)} \stackrel{\mathrm{d}}{=} C\mathbf{X} + \mathbf{D},$$

where $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ are independent copies of $\mathbf{X}$, $\stackrel{\mathrm{d}}{=}$ denotes equality in distributions. The vector $\mathbf{X}$ is called strictly stable if $\mathbf{D} = \mathbf{0}$. The vector $\mathbf{X}$ is symmetric stable if it is stable and

$$\mathrm{Prob}\{\mathbf{X} \in A\} = \mathrm{Prob}\{-\mathbf{X} \in A\}$$

for any Borel set A of $\mathbb{R}^d$. A random vector is α-stable if $C = (A^\alpha + B^\alpha)^{1/\alpha}$ where $0 < \alpha \leqslant 2$.

The characteristic function $\phi(\mathbf{k}) = \mathbb{E}\left[e^{i\langle \mathbf{k},\mathbf{X}\rangle}\right]$ of the α-stable vector $\mathbf{X} = (X_1, \ldots, X_d)$ in $\mathbb{R}^d$ is

$$\phi(\mathbf{k}) = \begin{cases} \exp\left\{-\int_{S_d} |\langle \mathbf{k},\mathbf{s}\rangle|^\alpha \left[1 - i\mathrm{sign}(\langle \mathbf{k},\mathbf{s}\rangle)\tan\frac{\pi\alpha}{2}\right]\Lambda(d\mathbf{s}) + i\langle \mathbf{k},\boldsymbol{\mu}^0\rangle\right\} \\ \quad \text{for} \quad \alpha \neq 1, \\ \exp\left\{-\int_{S_d} |\langle \mathbf{k},\mathbf{s}\rangle|^\alpha \left[1 + i\frac{2}{\pi}\mathrm{sign}(\langle \mathbf{k},\mathbf{s}\rangle)\ln(\langle \mathbf{k},\mathbf{s}\rangle)\right]\Lambda(d\mathbf{s}) + i\langle \mathbf{k},\boldsymbol{\mu}^0\rangle\right\} \\ \quad \text{for} \quad \alpha = 1, \end{cases}$$

where S_d is a unit sphere in $\mathbb{R}^d$ and $\Lambda(\cdot)$ is a spectral measure.

G. Samorodnitsky, and M. S. Taqqu, *Stable NonGaussian Random Processes*, (Chapman & Hall 1994).

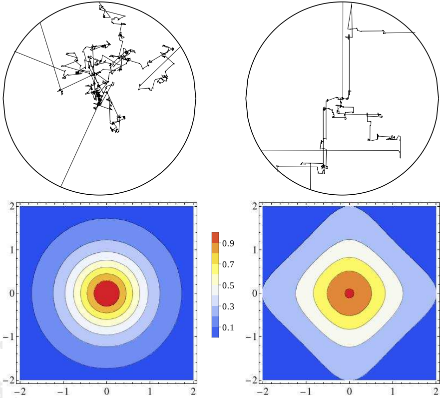
Cauchy distribution $\alpha=1$

For symmetric spectral measure concentrated on intersections of the axes with the unit sphere S_2 the bi-variate Cauchy ($\alpha = 1$) distribution is

$$p(x,y) = \frac{1}{\pi}\frac{\sigma}{(x^2+\sigma^2)} \times \frac{1}{\pi}\frac{\sigma}{(y^2+\sigma^2)}.$$

For continuous and uniform spectral measure

$$p(x,y) = \frac{1}{2\pi}\frac{\sigma}{(x^2+y^2+\sigma^2)^{3/2}}.$$



Equations in 1D

The Langevin equation

$$\frac{dx}{dt} = -V'(x) + \sigma\zeta_{\alpha,0}(t),$$

$$dx = -V'(x)dt + \sigma dL_{\alpha,0}(t)$$

is associated with the fractional Smoluchowski-Fokker-Planck equation

$$\begin{aligned}\frac{\partial p(x,t)}{\partial t} &= \frac{\partial}{\partial x}\left[V'(x)p(x,t)\right] + \sigma^{\alpha}\frac{\partial^{\alpha}p(x,t)}{\partial|x|^{\alpha}} \\ &= \frac{\partial}{\partial x}\left[V'(x)p(x,t)\right] - \sigma^{\alpha}(-\Delta)^{\alpha/2}p(x,t).\end{aligned}$$

The fractional Riesz-Weil derivative is defined via its Fourier transform

$$\mathcal{F}\left[\frac{\partial^{\alpha}p(x,t)}{\partial|x|^{\alpha}}\right] = \mathcal{F}\left[-(-\Delta)^{\alpha/2}p(x,t)\right] = -|k|^{\alpha}\mathcal{F}\left[p(x,t)\right].$$

P. D. Ditlevsen, Phys. Rev. E **60** 172 (1999).
D. Schertzer and M. Larchevêque, J. Duan, V. V. Yanowsky, S. Lovejoy, J. Math. Phys. **42** 200 (2001).

Equations in 1D

For $\alpha < 2$, and $V(x) = |x|^c$ stationary states exist for $c > 2 - \alpha$. Stationary states (if exist) have power-law asymptotics

$$p_{st}(x) \propto |x|^{-(c+\alpha-1)}.$$

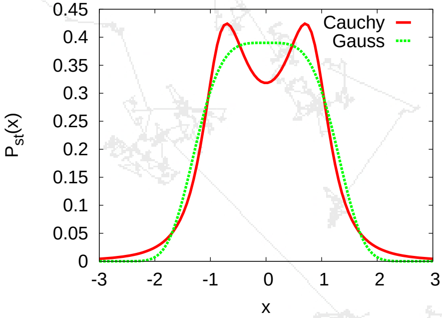
For $c = 2$ the stationary density is the same as the stable distribution associated with the underlying noise.

For $V(x) = \frac{1}{4}x^4$ and $\alpha = 1$

$$p_{st}(x) = \frac{\sigma}{\pi(\sigma^{4/3} - \sigma^{2/3}x^2 + x^4)}.$$

A. V. Chechkin, J. Klafter, V. Yu. Gonchar, R. Metzler and L. V. Tanatarov, Chem. Phys. **284** 233 (2002); Phys. Rev. E **67**, 010102 (2003).
B. Dybiec, I. M. Sokolov, A. V. Chechkin, J. Stat. Mech. P07008 (2010).

Stationary states (quartic – $V(x) = x^4/4$ – potential)



For $\alpha = 2$, the stationary states are of the Boltzmann-Gibbs type, i.e. $P(x) \propto \exp[-V(x)]$.

$$P_2(x) = \frac{\sqrt{2}}{\Gamma(\frac{1}{4})} \exp\left[-\frac{x^4}{4}\right].$$

For $\alpha < 2$, stationary solutions are no longer of the Boltzmann-Gibbs type. For $\alpha = 1$

$$P_1(x) = \frac{1}{\pi(x^4 - x^2 + 1)}.$$

A. V. Chechkin, J. Klafter, V. Yu. Gonchar, R. Metzler and L. V. Tanatarov, Chem. Phys. **284** 233 (2002); Phys. Rev. E **67**, 010102 (2003).

2D Langevin equation

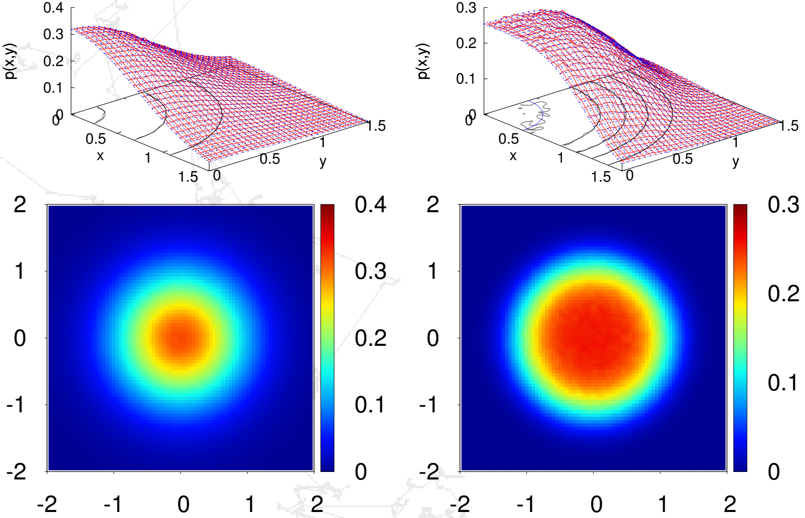
$$\frac{d\mathbf{r}}{dt} = -\nabla V(\mathbf{r}) + \sigma\zeta_\alpha(t),$$

$$d\mathbf{r} = -\nabla V(\mathbf{r})dt + \sigma d\mathbf{L}_\alpha(t).$$

Especially interesting potentials are

- harmonic: $V(x, y) = \frac{1}{2}r^2 = \frac{1}{2}(x^2 + y^2)$,
- quartic: $V(x, y) = \frac{1}{4}r^4 = \frac{1}{4}(x^2 + y^2)^2$.

Bivariate Gaussian



$V(x,y) = \frac{1}{2}(x^2 + y^2)$ (left panel) and $V(x,y) = \frac{1}{4}(x^2 + y^2)^2$ (right panel) subject to the bi-variate, uniform Gaussian white noise ($\alpha = 2$).

Equations in 2D

The associated Smoluchowski-Fokker-Planck equation

$$\frac{\partial p(\mathbf{r},t)}{\partial t} = \nabla \cdot [\nabla V(\mathbf{r}) p(\mathbf{r},t)] + \sigma^{\alpha} \mathbf{\Xi} p(\mathbf{r},t),$$

where $\mathbf{\Xi}$ is the fractional operator. $\nabla \cdot [\nabla V(\mathbf{r}) p(\mathbf{r},t)]$ originates due to the deterministic force $\mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r})$ acting on a test particle.

For the bi-variate α-stable noise with the uniform spectral measure the fractional operator

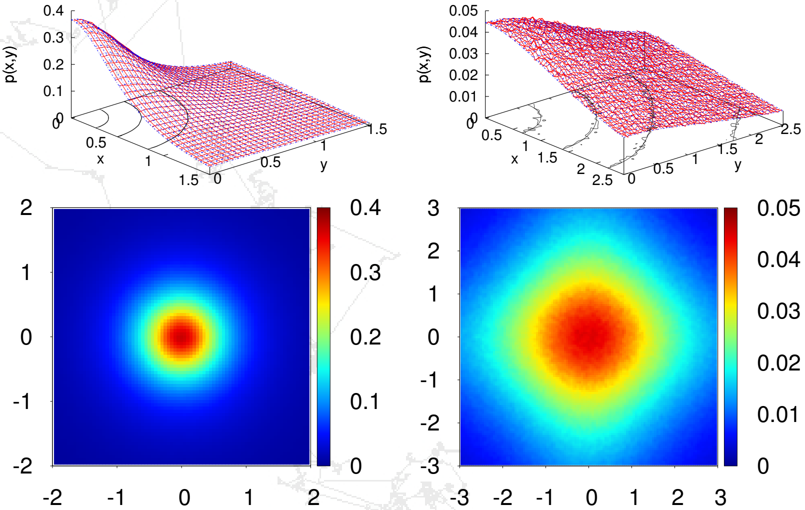
$$\mathbf{\Xi} = -(-\Delta)^{\alpha/2}.$$

For the bi-variate α-stable noise with the discrete symmetric spectral measure (located on intersections of S_2 with axis)

$$\mathbf{\Xi} = \frac{\partial^{\alpha}}{\partial |x|^{\alpha}} + \frac{\partial^{\alpha}}{\partial |y|^{\alpha}}.$$

S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications* (Gordon and Breach, Yverdon, 1993).
A. V. Chechkin, V. Y. Gonchar, and M. Szydlowski, Phys. Plasmas **9**, 78 (2002).

Bivariate Cauchy – parabolic potential



$V(x, y) = \frac{1}{2}(x^2 + y^2)$ with $\alpha = 1$ (Cauchy noise).

Bivariate Cauchy – parabolic potential – continuous $\Lambda(\cdot)$

Smoluchowski-Fokker-Planck equation

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}(xp) + \frac{\partial}{\partial y}(yp) - (-\Delta)^{\alpha/2} p.$$

In the Fourier space

$$\frac{\partial \hat{p}}{\partial t} = -k\frac{\partial \hat{p}}{\partial k} - l\frac{\partial \hat{p}}{\partial l} - (k^2+l^2)^{\alpha/2}\hat{p}.$$

The stationary density fulfills

$$k\frac{\partial \hat{p}}{\partial k} + l\frac{\partial \hat{p}}{\partial l} = -(k^2+l^2)^{\alpha/2}\hat{p},$$

$$(k^2+l^2)^{\alpha/2}\hat{p} + (k^2+l^2)^{1/2}\hat{p}' = 0,$$

where $\hat{p}' = \frac{\partial \hat{p}(\sqrt{k^2+l^2})}{\partial \sqrt{k^2+l^2}}$. The solution is

$$\hat{p} = \exp\left[-\frac{(k^2+l^2)^{\alpha/2}}{\alpha}\right].$$

Bivariate Cauchy – parabolic potential – discrete $\Lambda(\cdot)$

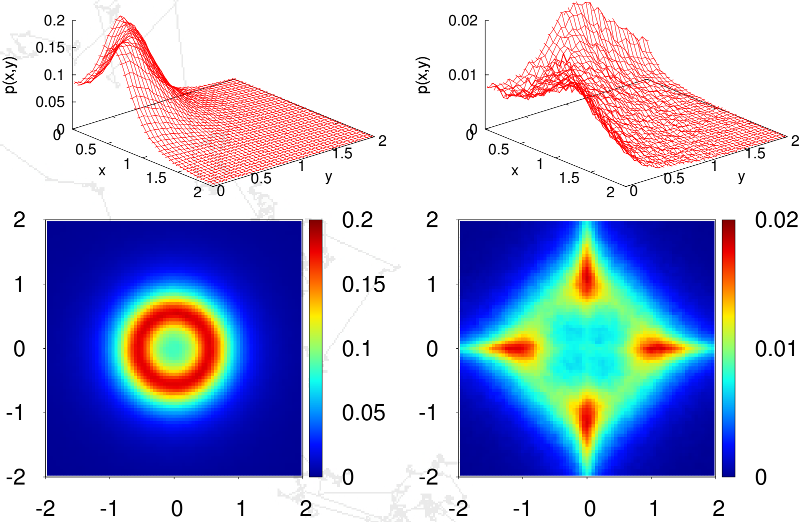
Smoluchowski-Fokker-Planck equation

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}(xp) + \frac{\partial}{\partial y}(yp) + \left(\frac{\partial^\alpha}{\partial |x|^\alpha} + \frac{\partial^\alpha}{\partial |y|^\alpha}\right) p.$$

In the Fourier space

$$\hat{p}(l)\left[k\frac{\partial \hat{p}(k)}{\partial k} + |k|^\alpha \hat{p}(k)\right] + \hat{p}(k)\left[l\frac{\partial \hat{p}(l)}{\partial l} + |l|^\alpha \hat{p}(l)\right] = 0. \quad (1)$$

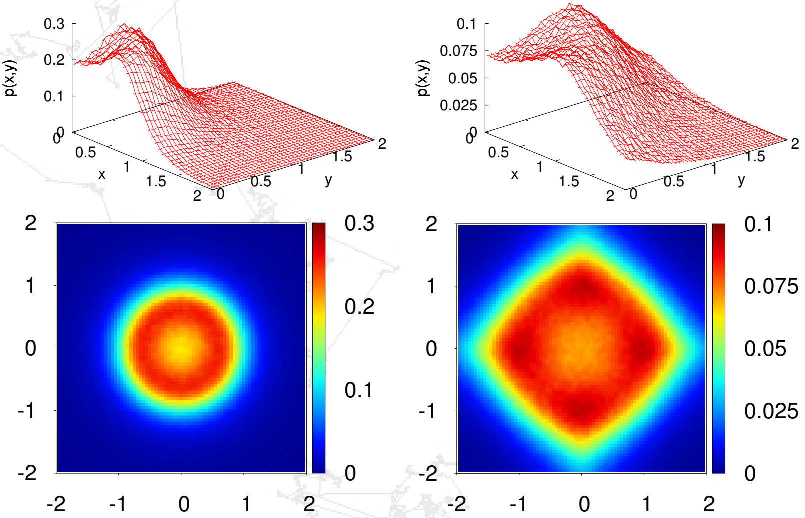
$\alpha = 0.5$ – quartic potential



$V(x,y) = \frac{1}{4}(x^2 + y^2)^2$ with $\alpha = 0.5$.

K. Szczepaniec and B. Dybiec, Phys. Rev. E **90**, 032128 (2014).

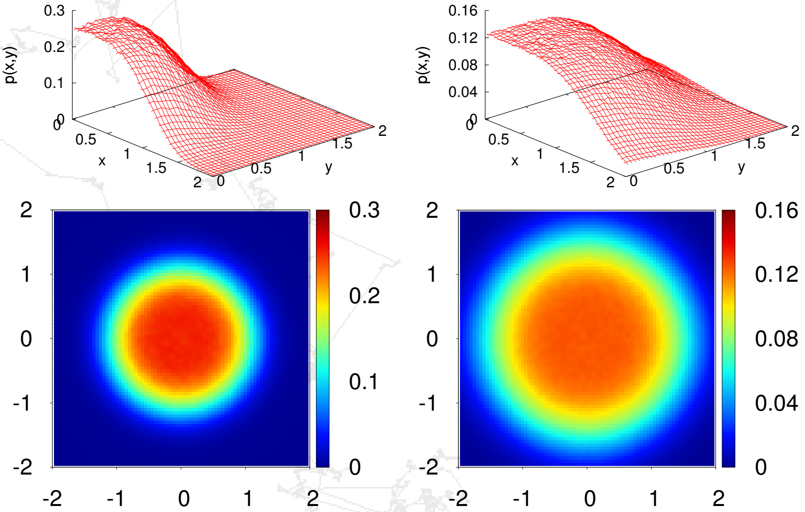
Bivariate Cauchy – quartic potential



$V(x,y) = \frac{1}{4}(x^2 + y^2)^2$ with $\alpha = 1$.

K. Szczepaniec and B. Dybiec, Phys. Rev. E **90**, 032128 (2014).

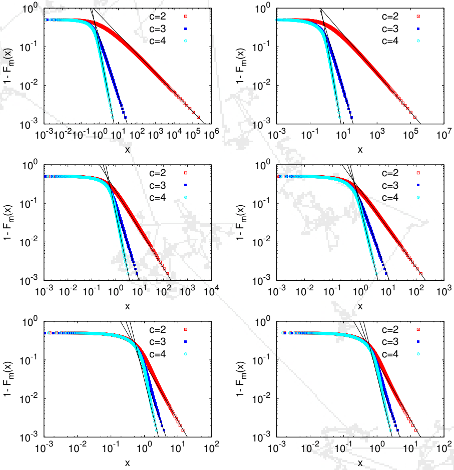
$\alpha = 1.9$ – quartic potential



$V(x, y) = \frac{1}{4}(x^2 + y^2)^2$ with $\alpha = 1.9$.

K. Szczepaniec and B. Dybiec, Phys. Rev. E **90**, 032128 (2014).

Marginal densities



Survival probabilities, $S(x) = 1 - F_m(x)$, for marginal densities of x for uniform (left panel) and symmetric discrete (right panel) spectral measures. α: $\alpha = 0.5$ (top row), $\alpha = 1$ (middle row) and $\alpha = 1.5$ (bottom row). Potentials of $V(x, y) = (x^2 + y^2)^{c/2}$ type: harmonic ($c = 2$), cubic ($c = 3$) and quartic ($c = 4$). Solid lines present $x^{-(c+\alpha-2)}$ power-law asymptotics of survival probailities.

K. Szczepaniec and B. Dybiec, Phys. Rev. E **90**, 032128 (2014).

1D noise and 2D phase space

Damped (Brownian) harmonic oscillator

Damped harmonic oscillator

$$\ddot{x}(t) = -\gamma \dot{x}(t) - V'(x) + \xi(t).$$

Klein-Kramers equation

$$\frac{\partial P}{\partial t} = -v\frac{\partial P}{\partial x} + \frac{\partial}{\partial v}\left\{\left[V'(x) + \gamma v\right] P\right\} + D_2 \frac{\partial^2 P}{\partial v^2},$$

where

$$P = P(x, v; t | x_0, v_0; t_0).$$

Stationary solution

$$p(x, v) \propto \exp\left[-\frac{V(x)}{k_B T} - \frac{mv^2}{2k_B T}\right]$$

→ x and v are independend random variables
→ for $V(x) \propto x^2$ equipartition theorem

Damped (Lévy) harmonic oscillator

Equation of motion

$$\ddot{x}(t) = -\gamma\dot{x}(t) - V'(x) + \zeta(t),$$

can be rewritten as

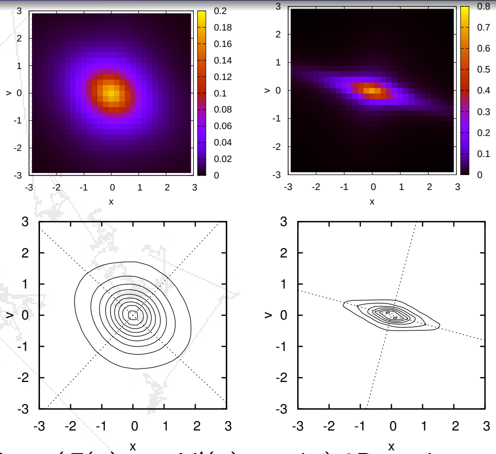
$$\left\{ \begin{array}{lll} \frac{dx}{dt} & = & v \\ \frac{dv}{dt} & = & -\gamma v - V'(x) + \zeta(t) \end{array} \right. .$$

Fractional Kleina-Kramers equation

$$\frac{\partial P}{\partial t} = -v\frac{\partial P}{\partial x} + \frac{\partial}{\partial v}\left\{\left[V'(x) + \gamma v\right] P\right\} + D_\alpha \frac{\partial^\alpha P}{\partial |v|^\alpha},$$

where

$$P = P(x, v; t|x_0, v_0; t_0).$$



Due to linear force ($F(x) = -V'(x) = -kx$) 2D random variable (x, v) is a 2D α-stable variable:

- for $0 < \alpha < 2$ x and v are not independent,
- there is no equipartition theorem.

I. M. Sokolov, B. Dybiec and W. Ebelling, Phys. Rev. E **83**, 041118 (2011).

Conclusions

2D systems driven by bi-variate α-stable noises display analogous universlities like 1D systems.

Thank you very much for your attention!!

- I. M. Sokolov, B. Dybiec and W. Ebelling, Phys. Rev. E **83**, 041118 (2011); also arXiv:1010.2657
- K. Szczepaniec and B. Dybiec, *Stationary states in 2D systems driven by bi-variate Lévy noises*, Phys. Rev. E **90**, 032128 (2014); also arXiv:1406.7103.
- K. Szczepaniec and B. Dybiec, *Resonant activation in 2D and 3D systems driven by multi-variate Lévy noises*, J. Stat. Mech. P09022 (2014); also arXiv:1406.7810.
- K. Szczepaniec and B. Dybiec, *Escape from bounded domains driven by multivariate -stable noises*, J. Stat. Mech. P06031 (2015); also arXiv:1406.7810.
- B. Dybiec and K. Szczepaniec, *Escape from hypercube driven by multi-variate -stable noises: role of independence*, Eur. Phys. J. B **88**, 184 (2015).

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